Unimodular relativity and cosmological constant : Comments

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We show that the conclusion that matter stress-energy tensor satisfies the usual covariant continuity law, and the cosmological constant is still a constant of integration arrived at by Finkelstein et al (42, 340, 2001) is not valid.

A recent paper [1] reports the status of dark matter from the view point of observational cosmology. It is also noted that anisotropic cosmic microwave background radiation provides 'compelling argument for a non-zero cosmological constant'. Theoretically, cosmological constant was first introduced by Einstein, but later he felt that this term reduced the 'logical simplicity' of the theory [2]. The interested reader may find extensive references to the literature in the reviews [3]. In this brief comment, we re-examine the problem of energy-momentum conservation law in unimodular gravity. Anderson and Finkelstein [4] envisage a cellular structure of space-time, and develop unimodular theory of relativity based on an action principle for a measure manifold with a fundamental measure $\mu(x)$. The unimodular condition is

$$\sqrt{-g}d^4x = \mu(x)d^4x \tag{1}$$

To derive the field equations based on the action principle, the measure is assumed to be a fixed nondynamical field. Choosing $\mu(x) = 1$ may give rise to a simpler unimodular coordinate choice. The unimodular condition (1) is incorporated in the action function using Lagrange's undetermined multiplier, $\lambda(x)$. The field equations derived from the variational principle admit a cosmological term, and the cosmological constant is an integration constant. In [5] we point out that without an additional assumption on the covariant divergence of the matter energy-momentum tensor. $T^{\mu\nu}$, the cosmological constant is not 'constant'. In a recent paper, Finkelstein et al [6] elucidate unimodular relativity emphasizing the role of a conformal metric tensor $f_{\mu\nu}-$ 'the sole gravitational variable of unimodular relativity'.

We believe the approach based on $f_{\mu\nu}$ may have interesting physics, specially due to the possibility of exploring Weyl geometry in this framework. Here we focus our attention on the consequences of ambiguous extended action, S' presented in Sec. III of [6]. We use the notations of the authors [6], and give main steps in the following. The matter Lagrangian density is

$$L'_M = L_M + \Delta_M L \tag{2}$$

The ambiguity is assumed to be of the form

$$\Delta_M L = \left[\frac{\mu(x)}{\sqrt{-g}} - 1\right] l_M \tag{3}$$

where l_M is a function of matter field variables and $g_{\mu\nu}$. The Lagrange multiplier $\lambda(x)$ is introduced to incorporate the unimodular condition (1), and the action integral S' is constructed as usual, see eqn. (13) in [6]. Varying $g_{\mu\nu}$ gives the field equation

$$G^{\mu\nu} - \frac{\lambda}{2} g^{\mu\nu} = 8\pi G T'^{\mu\nu} \tag{4}$$

Here $G^{\mu\nu}$ is the Einstein tensor, and the ambiguous energy-momentum tensor is

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\Delta_M L)}{\delta g_{\mu\nu}} \tag{5}$$

It is straightforward to calculate λ taking trace of (4)

$$\lambda = -\frac{8\pi G T' + R}{2} \tag{6}$$

The field equation (4) reduces to

$$R_{\mu\nu} - \frac{R g_{\mu\nu}}{4} = 8\pi G (T'_{\mu\nu} - \frac{g_{\mu\nu} T'}{4}) \tag{7}$$

If the ambiguity in energy-momentum tensor is calculated following the prescription (3), we get

$$\Delta T^{\mu\nu} = \frac{g^{\mu\nu} l_M}{2} \tag{8}$$

Evidently there is an error in eqns. (24) and (25) of [6]. Using (8), and taking trace of eqn. (4), we obtain

$$2\lambda = -R - 8\pi G(T + 2 l_M) \tag{9}$$

Both λ and l_M get eliminated in the final field equation

$$R_{\mu\nu} - \frac{R g_{\mu\nu}}{4} = 8\pi G (T_{\mu\nu} - \frac{g_{\mu\nu} T}{4}) \tag{10}$$

Finkelstein et al [6] note that $T'_{\mu\nu}$ is not covariantly continuous in unimodular relativity and state that that seems to justify modified covariant divergence law, eqn(7) of [5]. However, the authors assert that $T_{\mu\nu}$ satisfies the usual covariant divergence law, and the cosmological constant is still a constant of integration. Though authors do not state it explicitly,the additive ambiguity in $T_{\mu\nu}$, and the discussion following eqn(23) in their paper seem to imply that somehow the ambiguity in the matter field Lagriangian leads to this result. Does this result that usual covariant continuity law holds for $T_{\mu\nu}$ follow from their theory?

To analyze this question first we make a remark based on [5]. In the notation of that paper eqn(1) and eqn(7) of [5] give L to be a constant using the Bianchi identity. From eqn(6) of that paper it follows that R+T is constant. One could assume the constancy of R+T and infer the covarint continuity of the energy-momentum tensor. Here also we use the Bianchi identity and derive the covariant divergence law for $T\mu\nu$ in the Finkelstein et al theory. Taking covariant divergence of eqn(4) above we get

$$8\pi G T'^{\mu\nu}{}_{:\nu} = -\frac{1}{2} g^{\mu\nu} \lambda_{\nu} \tag{11}$$

From eqn(5), it is straightforward to calculate

$$8\pi G T^{\mu\nu}{}_{:\nu} = -\frac{1}{2} g^{\mu\nu} (\lambda + 8\pi G l_M)_{,\nu} \tag{12}$$

Evidently $T_{\mu\nu}$ is not covariantly continuous as shown by eqn(12). Though l_M does not appear in the field eqns (7) and (10),from expression(9) as well as eqn(12) it becomes clear that l_M mofifies the cosmological constant term. Let us denote it by $\lambda_{eff}(= \lambda + 8\pi G l_M)$.

If the consistency of the theory given by Finkelstein et al has to be maintained then following conclusions are inevitable:

1. If we impose the condition that the covariant divergence of $T_{\mu\nu}$ vanishes then λ_{eff} is constant. From eqn(9) it follows that $R + 8\pi GT$ is also constant. However $T'\mu\nu$ still satisfies the modified covariant divergence law, eqn(11) and λ is not necessarily a constant. Imposing the further condition that covariant divergence of $T'_{\mu\nu}$ is also zero, λ too becomes a constant. As a consequence $R + \pi GT'$ as well as l_M also become constant.

2. Alternatively, imposing the condition that $R+8\pi GT$ and $R+8\pi GT'$ are constant,the constancy of λ and l_M as well as the covariant continuity of both primed and unprimed energy-momentum tensors follow.

3. The principal result of [6] therefore is an assumption, not a consequence of the theory.

We may point out that it is also inconsistent to allow cosmological constant to be a field variable and simultaneously require covariant continuity of energy-momentum tensor as is done by Ng and van Dam [7]. To end the paper, let us first note a mathematical result:both λ and l_M are variable such that the sum $\lambda + 8\pi G l_M$ is required to be zero. In that case eqn(4) reduces to the standard Einstein field equation without cosmological constant. Though the physical significance of ambiguity introduced in [6] is not clear, it seems interesting to speculate that λ and l_M in eqn(9) correspond to geometric and vacuum energy aspects of the unimodular world. Does that mean that in the standard theory with Einstein field equation both contributions cancel each other under this condition ? We leave the answer to this question for future investigations.

I am grateful to Profs. D. Finkelstein and A.A.Galiutdinov for extensive e-correspondence on this subject. Library facility of Banaras Hindu University, Varanasi is acknowledged.

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